STRATIFIED MORSE THEORY IN ARRANGEMENTS

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For Robert MacPherson on the occasion of his sixtieth birthday.

ABSTRACT. This paper is a survey of our work based on the stratified Morse theory of Goresky and MacPherson. First we discuss the Morse theory of Euclidean space stratified by an arrangement. This is used to show that the complement of a complex hyperplane arrangement admits a minimal cell decomposition. Next we review the construction of a cochain complex whose cohomology computes the local system cohomology of the complement of a complex hyperplane arrangement. Then we present results on the Gauss-Manin connection for the moduli space of arrangements of a fixed combinatorial type in rank one local system cohomology.

1. INTRODUCTION

Let V be a complex vector space of dimension $\ell \geq 1$. A hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a set of $n \geq 0$ hyperplanes in V. Let $\mathsf{M} = V \setminus \bigcup_{j=1}^n H_j$ denote the complement. Introduce coordinates $u_1, \ldots, u_\ell$ in V and, for each j, $1 \leq j \leq n$, choose a degree one polynomial α_j so that the hyperplane $H_j \in \mathcal{A}$ is defined by the vanishing of α_j. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a set of complex weights for the hyperplanes. Given $\boldsymbol{\lambda}$, we define a multivalued holomorphic function on M by

$$\Phi(u; \boldsymbol{\lambda}) = \prod_{j=1}^n \alpha_j^{\lambda_j}.$$

A generalized hypergeometric integral is of the form

$$\int_\sigma \Phi(u; \boldsymbol{\lambda})\eta$$

where σ is a suitable domain of integration and η is a holomorphic form on M, see [AK]. When $\ell = 1$, $n = 3$ and $\alpha_1 = u, \alpha_2 = u - 1, \alpha_3 = u - x$, this is the Gauss hypergeometric integral. Selberg's integral [Se] is another special case:

$$\int_0^1 \cdots \int_0^1 (u_1 \cdots u_\ell)^{x-1}[(1-u_1)\cdots(1-u_\ell)]^{y-1}|\Delta(u)|^{2z}\, du_1 \ldots du_\ell$$

where $\Delta(u) = \prod_{i<j}(u_j - u_i)$. Hypergeometric integrals occur in the representation theory of Lie algebras and quantum groups [SV2, V]. In physics, these hypergeometric integrals form solutions to the Knizhnik-Zamolodchikov differential equations in conformal field theory [SV1, V]. The space of integrals is identified

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with a cohomology group, $H^\ell(\mathsf{M};\mathcal{L})$, of the complement, with coefficients in a complex rank one local system. Associated to $\boldsymbol{\lambda}$, there is a rank one representation $\rho : \pi_1(\mathsf{M}) \to \mathbb{C}^*$, given by $\rho(\gamma_j) = t_j$, where $\mathbf{t} = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$ is defined by $t_j = \exp(-2\pi\,\mathrm{i}\,\lambda_j)$, and γ_j is any meridian loop about the hyperplane H_j of $\mathcal{A}$, and a corresponding rank one local system $\mathcal{L} = \mathcal{L}_{\mathbf{t}} = \mathcal{L}_{\boldsymbol{\lambda}}$ on M. Equivalently, weights $\boldsymbol{\lambda}$ determine a flat connection on the trivial line bundle over M, with connection one-form $\omega_{\boldsymbol{\lambda}} = d\log\Phi(u;\boldsymbol{\lambda})$.

The first problem is to calculate the local system cohomology groups $H^q(\mathsf{M},\mathcal{L})$. The methods used by Aomoto and Kita [AK], Esnault, Schechtman, and Viehweg [ESV], Schechtman, Terao, and Varchenko [STV] and others are described in detail in [OT2]. These use the twisted de Rham complex, $(\Omega^\bullet(*\mathcal{A}),\nabla)$, of global rational differential forms on V with arbitrary poles along the divisor $\bigcup_{j=1}^n H_j$, with differential $\nabla(\eta) = d\eta + \omega_{\boldsymbol{\lambda}} \wedge \eta$. The cochain groups of this complex are infinite dimensional. Conditions must be imposed on the weights in order to reduce the problem to a finite dimensional setting. These are the nonresonance conditions of [STV]. Under these conditions, the calculation may be reduced to combinatorics and yields $H^q(\mathsf{M};\mathcal{L}) = 0$ for $q \neq \ell$ and $\dim H^\ell(\mathsf{M};\mathcal{L}) = |e(\mathsf{M})|$, where $e(\mathsf{M})$ is the Euler characteristic of the complement. This approach provides less information for resonant weights, those for which the aformentioned nonresonance conditions do not hold. By contrast, the results obtained below using stratified Morse theory are valid for arbitrary weights.

Weights $\boldsymbol{\lambda}$ give rise to a local system on the complement of every arrangement that is combinatorially equivalent to $\mathcal{A}$. The resulting local system cohomology groups comprise a flat vector bundle over the moduli space of such arrangements. The second problem is to determine the Gauss-Manin connection in this cohomology bundle. For instance, the Gauss hypergeometric function is defined on the complement of the arrangement of three points in $\mathbb{C}$. It satisfies a second order differential equation which, when converted into a system of two linear differential equations, may be interpreted as a Gauss-Manin connection on the moduli space of arrangements of the same combinatorial type [OT2]. This idea has been generalized to all arrangements by Aomoto [A2] and Gelfand [G]. The connection is obtained by differentiating in the moduli space. For arrangements in general position, and nonresonant weights, explicit connection matrices were obtained by Aomoto and Kita [AK]. Unfortunately, this pioneering work is available only in Japanese. The relevant matrices have been reproduced in [OT2, CO3].

The (flat) Gauss-Manin connection in the cohomology bundle corresponds to a representation of the fundamental group of the moduli space. The endomorphisms arising in the connection one-form, which we refer to as Gauss-Manin endomorphisms, may be realized as logarithms of certain automorphisms. This interpretation, used in [CO4], allows for local calculations, valid for all arrangements and all weights. This paper is a survey of our work on these problems.

Section 2 presents basic results on the Morse theory of Euclidean space stratified by an arrangement (of subspaces), following [GM, C1]. In Section 3, we use these results to give a proof of a theorem of Dimca and Papadima [DP1] and Randell [Ra2], which asserts that the complement of a complex hyperplane arrangement is a minimal space, admiting a cell decomposition for which the number of q-cells is equal to the q-th Betti number for each q. In Section 4, we review the stratified Morse theory construction from [C1, CO1] of a finite cochain complex $(K^\bullet(\mathcal{A}),\Delta^\bullet)$,

the cohomology of which is naturally isomorphic to $H^*(\mathsf{M};\mathcal{L})$. This leads to the construction of the universal complex $(\mathsf{K}^\bullet,\Delta^\bullet(\mathbf{x}))$ for local system cohomology, where $\mathsf{K}^q=\Lambda\otimes_{\mathbb{C}}K^q$ and $\Lambda=\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$. We recall a combinatorial model for $H^*(\mathsf{M},\mathbb{C})$, called the Orlik-Solomon algebra, $A(\mathcal{A})$. The one-form $\omega_{\boldsymbol\lambda}$ corresponds to an element $a_{\boldsymbol\lambda}$ of the Orlik-Solomon algebra. Multiplication by $a_{\boldsymbol\lambda}$ gives this algebra the structure of a cochain complex, $(A^\bullet(\mathcal{A}),a_{\boldsymbol\lambda})$. The Aomoto complex is the universal complex for this cochain complex. It is chain equivalent to the linearization of the universal complex. This informs on the relationship between the characteristic varieties of complements of arrangements (jumping loci for local system cohomology) and the resonance varieties of arrangements (jumping loci for the cohomology of the Orlik-Solomon complex).

In Section 5, we move from consideration of a fixed arrangement to the study of all arrangements of a given combinatorial type. We define the moduli space of arrangements with a fixed combinatorial type $\mathcal{T}$ and the set $\mathrm{Dep}(\mathcal{T})$ of dependent sets in type $\mathcal{T}$. We present results concerning the homology of the moduli space.

In Section 6, we work with a smooth, connected component of the moduli space, $\mathsf{B}(\mathcal{T})$. There is a fiber bundle $\pi:\mathsf{M}(\mathcal{T})\to\mathsf{B}(\mathcal{T})$ whose fibers, $\pi^{-1}(\mathsf{b})=\mathsf{M}_{\mathsf{b}}$, are complements of arrangements $\mathcal{A}_{\mathsf{b}}$ of type $\mathcal{T}$. Since $\mathsf{B}(\mathcal{T})$ is connected, M_{b} is diffeomorphic to M. The fiber bundle $\pi:\mathsf{M}(\mathcal{T})\to\mathsf{B}(\mathcal{T})$ is locally trivial. Given a local system on the fiber, consider the associated flat vector bundle $\pi^q:\mathbf{H}^q(\mathcal{L})\to\mathsf{B}(\mathcal{T})$, with fiber $(\pi^q)^{-1}(\mathsf{b})=H^q(\mathsf{M}_{\mathsf{b}};\mathcal{L}_{\mathsf{b}})$ at $\mathsf{b}\in\mathsf{B}(\mathcal{T})$ for each q, $0\le q\le\ell$. Fixing a basepoint $\mathsf{b}\in\mathsf{B}(\mathcal{T})$, the operation of parallel translation of fibers over curves in $\mathsf{B}(\mathcal{T})$ in the vector bundle $\pi^q:\mathbf{H}^q(\mathcal{L})\to\mathsf{B}(\mathcal{T})$ provides a complex representation

$$\Psi^q_{\mathcal{T}}:\pi_1(\mathsf{B}(\mathcal{T}),\mathsf{b})\longrightarrow\mathrm{Aut}_{\mathbb{C}}(H^q(\mathsf{M}_{\mathsf{b}};\mathcal{L}_{\mathsf{b}})).$$

The loops of primary interest are those linking moduli spaces of codimension one degenerations of $\mathcal{T}$. Such a degeneration is a type $\mathcal{T}'$ whose moduli space $\mathsf{B}(\mathcal{T}')$ has codimension one in the closure of $\mathsf{B}(\mathcal{T})$. In this case we say that $\mathcal{T}$ covers $\mathcal{T}'$.

When $\mathcal{T}$ covers $\mathcal{T}'$ and $\gamma\in\pi_1(\mathsf{B}(\mathcal{T}),\mathsf{b})$ is a simple loop linking $\mathsf{B}(\mathcal{T}')$ in $\overline{\mathsf{B}(\mathcal{T})}$, write $\Psi^q_{\mathcal{T}}(\gamma)=\exp(-2\pi\,\mathrm{i}\,\Omega)$. We denote this Gauss-Manin endomorphism in the bundle $\pi^q:\mathbf{H}^q(\mathcal{L})\to\mathsf{B}(\mathcal{T})$ by $\Omega=\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'),\mathsf{B}(\mathcal{T}))$. The rest of this survey reports on our results concerning these endomorphisms.

In Section 7, for each subset S of hyperplanes, we define an endomorphism $\tilde\omega^\bullet_S$ of the Aomoto complex of a general position arrangement of n hyperplanes in $\mathbb{C}^\ell$. When $\mathcal{T}$ covers $\mathcal{T}'$, we construct a suitable linear combination of these maps, which induces an endomorphism of the Aomoto complex of type $\mathcal{T}$. The specialization $\mathbf{y}\mapsto\boldsymbol\lambda$ in the Aomoto complex then yields an endomorphism $\omega^\bullet_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})$ of the Orlik-Solomon complex $A^\bullet(\mathcal{T})=A^\bullet(\mathcal{A})$ of an arrangement $\mathcal{A}$ of type $\mathcal{T}$. This leads to our main result, stated in more detail in Section 7.

Theorem ([CO4])**.** *Let M be the complement of an arrangement $\mathcal{A}$ of type $\mathcal{T}$ and let $\mathcal{L}$ be the local system on M defined by weights $\boldsymbol\lambda$. Suppose $\mathcal{T}$ covers $\mathcal{T}'$. Then there is a surjection $\xi^q:A^q(\mathcal{T})\twoheadrightarrow H^q(\mathsf{M},\mathcal{L})$ so that the Gauss-Manin endomorphism $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'),\mathsf{B}(\mathcal{T}))$ in local system cohomology is determined by the equation*

$$\xi^q\circ\omega^q_{\boldsymbol\lambda}(\mathcal{T}',\mathcal{T})=\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'),\mathsf{B}(\mathcal{T}))\circ\xi^q.\quad\square$$

In Section 8, we report on the spectrum of the Gauss-Manin endomorphism. The pair $(\mathcal{T}',\mathcal{T})$ determines a set of hyperplanes $S\subset\mathcal{A}$ and an integer r. We call (S,r) the principal dependence. Let $\lambda_S=\sum_{H_j\in S}\lambda_j$.

Theorem ([CO5])**.** *Suppose $\mathcal{T}$ covers $\mathcal{T}'$ with principal dependence (S, r). Let $\boldsymbol{\lambda}$ be a collection of weights satisfying $\lambda_S \neq 0$. Then the Gauss-Manin endomorphism $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$ is diagonalizable, with spectrum contained in $\{0, \lambda_S\}$.* □

We illustrate these results with an example in Section 9.

2. Morse Functions for Arrangements

Goresky and MacPherson developed stratified Morse theory in order to extend the class of spaces to which Morse theory applies. This generalization may be used to study singular spaces, noncompact spaces, etc. The latter is illustrated in Part III of their book [GM] using real subspace arrangements. The topology of the complement of such an arrangement is analyzed by Morse theoretic means, by considering the stratification of the ambient Euclidean space determined by the arrangement and realizing the complement as one of the strata. We recall some of their constructions and results needed in the sequel.

Let V be a real vector space of dimension $\ell \geq 1$, and let $\mathcal{A}$ be an arrangement of affine subspaces in V. An edge (or flat) of $\mathcal{A}$ is a nonempty intersection X of elements of $\mathcal{A}$. Let $L = L(\mathcal{A})$ be the set of all edges of $\mathcal{A}$. Unless otherwise noted, we partially order the set L by reverse inclusion.

The arrangement $\mathcal{A}$ gives rise to a Whitney stratification $\mathcal{S}$ of V with a stratum

$$\mathcal{S}_X = X \setminus \bigcup_{Y \subsetneq X} Y$$

for each edge $X \in L$. The complement M of $\mathcal{A}$ is the stratum corresponding to the edge V (the intersection of no elements of $\mathcal{A}$). For any edge X, the closure of $\mathcal{S}_X$ is X. Note that a complex hyperplane arrangement may be viewed as a real subspace arrangement with even-dimensional strata.

For almost any point $p \in \mathsf{M}$, the function $f : V \to \mathbb{R}$ given by

$$f(u) = [\text{distance}(p, u)]^2 \tag{2.1}$$

is a Morse function on V with respect to the stratification $\mathcal{S}$, see [GM, I.2.2]. For $r \in \mathbb{R}$, let

$$\mathsf{M}_{\leq r} = \{u \in \mathsf{M} \mid f(u) \leq r\}.$$

The function f has a unique critical point on each edge. It is a minimum. Furthermore, Goresky and MacPherson show in [GM, III.3] that the Morse function f is *perfect*: if $v \in \mathbb{R}$ is a critical value and $\epsilon > 0$ is sufficiently small, the long exact homology sequence of the pair $(\mathsf{M}_{\leq v+\epsilon}, \mathsf{M}_{\leq v-\epsilon})$ splits into short exact sequences

$$0 \to H_q(\mathsf{M}_{\leq v-\epsilon}; \mathbb{Z}) \to H_q(\mathsf{M}_{\leq v+\epsilon}; \mathbb{Z}) \to H_q(\mathsf{M}_{\leq v+\epsilon}, \mathsf{M}_{\leq v-\epsilon}; \mathbb{Z}) \to 0. \tag{2.2}$$

Using this, they calculate the homology $H_*(\mathsf{M}; \mathbb{Z})$ in terms of the poset L (ordered by inclusion), see [GM, III.1.3. Theorem A]. This result has prompted a great deal of work on the cohomology of the complement of a subspace arrangement, culminating with the determination of the cup product structure of this cohomology ring by de Longeville and Schultz [dLS] and Deligne, Goresky, and MacPherson [DGM].

One can produce a Morse function such as (2.1) that meets the strata of V according to codimension.

Definition 2.1. *Let Z be a Whitney stratified subset of Euclidean space. A Morse function $f : Z \to \mathbb{R}$ is said to be* weakly self-indexing *with respect to the stratification $\{S_\alpha\}$ of Z if for each q, $0 \le q \le \dim Z$, we have*

$$\max_{\operatorname{codim} S_\alpha = q-1} \{\textit{critical values of } f \mid S_\alpha\} < \min_{\operatorname{codim} S_\beta = q} \{\textit{critical values of } f \mid S_\beta\}.$$

Proposition 2.2. *Let $\mathcal{A}$ be an arrangement of subspaces in the real vector space V. Then there is a positive definite quadratic form $f : V \to \mathbb{R}$ which is a weakly self-indexing Morse function with respect to the stratification $\{S_\alpha\}$ of V given by $\mathcal{A}$, whose critical points consist of a unique minimum on each stratum.* $\square$

The proof of this result given in [C1, §1] shows that there are choices of coordinates $\{u_i\}$ on V and positive constants $\{\omega_i\}$, $1 \le i \le \ell = \dim V$, for which the quadratic form $f(u_1, u_2, \dots, u_\ell) = \sum_{i=1}^{\ell} \omega_i u_i^2$ is a weakly self-indexing Morse function with respect to the stratification determined by $\mathcal{A}$. This provides an inductive algorithm for the construction of a complete flag in V that is transverse to the arrangement $\mathcal{A}$.

In the rest of this paper, we return to the special case of a complex hyperplane arrangement where we can say more.

3. Minimality

The notion of *minimality* has played a significant role in recent work on the topology of arrangements, see for instance the work of Papadima and Suciu [PS], Dimca and Papadima [DP1, DP2], and Randell [Ra2].

Definition 3.1. *A space X is said to be* minimal *if X has the homotopy type of a connected, finite-type CW-complex W such that, for each $q \ge 0$, the number of q-cells in W is equal to the rank of $H_q(\mathsf{X};\mathbb{Z})$.*

Note that, for a minimal space X, all homology groups $H_q(\mathsf{X};\mathbb{Z})$ are finitely generated and torsion-free. If X is a 1-connected space with the homotopy type of a connected, finite-type CW-complex, and the homology of X is torsion-free, then X is minimal by work of Anick [An]. However, many spaces (with torsion-free homology) are not minimal. For instance, the complement of a non-trivial knot does not admit a minimal cell decomposition.

Dimca and Papadima [DP1] and Randell [Ra2] used various forms of Morse theory to show that the complement of a complex hyperplane arrangement is minimal. This result may also be established using stratified Morse theory.

Theorem 3.2. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a complex hyperplane arrangement in the complex vector space $V \cong \mathbb{C}^\ell$. Then the complement $\mathsf{M} = \mathsf{M}(\mathcal{A}) = V \setminus \bigcup_{i=1}^{n} H_i$ is a minimal space.*

Proof. Without loss of generality, assume that $\mathcal{A}$ is an *essential* arrangement in $\mathbb{C}^\ell$, that is, $\mathcal{A}$ contains ℓ linearly independent hyperplanes. Then the edges of $\mathcal{A}$ have codimensions 0 through ℓ. The proof is by induction on ℓ.

In the case $\ell = 1$, $\mathcal{A}$ is a finite collection of points in $V = \mathbb{C}$, and the complement $\mathsf{M}(\mathcal{A})$ has the homotopy type of a bouquet of circles, which is a minimal space.

For general ℓ, let

$$\mathcal{F}: \quad \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots \subset \mathcal{F}^{\ell-1} \subset \mathcal{F}^\ell = V, \tag{3.1}$$

be a complete flag in $V = \mathbb{C}^\ell$ that is transverse to the Whitney stratification of V determined by $\mathcal{A}$. Choose coordinates $\{u_i\}$ so that, for each $k \le \ell - 1$, $\mathcal{F}^k = \{u_{k+1} = \cdots = u_\ell = 0\}$. Let $f : V \to \mathbb{R}$ be a Morse function "about" the flag $\mathcal{F}$ that is weakly self-indexing with respect to the stratification of V determined by $\mathcal{A}$.

Since f is weakly self-indexing, there are constants a and b so that all critical values of f on edges of codimension less than ℓ are smaller than a, and all critical values of f on edges of codimension ℓ are in the interval (a, b). For such a and b, $\mathsf{M}_{\le b}$ is a deformation retract of the complement M of $\mathcal{A}$, and $\mathsf{M} \cap \mathcal{F}^{\ell-1}$ is a deformation retract of $\mathsf{M}_{\le a}$. Since $\mathsf{M} \cap \mathcal{F}^{\ell-1}$ is the complement of the arrangement $\mathcal{A} \cap \mathcal{F}^{\ell-1}$ in $\mathcal{F}^{\ell-1} = \mathbb{C}^{\ell-1}$, by induction, $\mathsf{M} \cap \mathcal{F}^{\ell-1} \simeq \mathsf{M}_{\le a}$ is a minimal space. So it suffices to show that M has the homotopy type of a space obtained from $\mathsf{M} \cap \mathcal{F}^{\ell-1}$ by attaching $b_\ell(\mathsf{M})$ ℓ-cells, where $b_k(\mathsf{M}) = \operatorname{rank} H_k(\mathsf{M}; \mathbb{Z})$ denotes the k-th Betti number of M.

By the Lefschetz hyperplane section theorem of Hamm and Lê [HL] (see also [GM]), M is obtained from $\mathsf{M} \cap \mathcal{F}^{\ell-1}$ by attaching at least $b_\ell(\mathsf{M})$ ℓ-cells, and the number of ℓ-cells is equal to the rank of the homology group $H_\ell(\mathsf{M}, \mathsf{M} \cap \mathcal{F}^{\ell-1})$. Using the fact that the Morse function f is perfect repeatedly, we see that the long exact sequence of the pair $(\mathsf{M}, \mathsf{M} \cap \mathcal{F}^{\ell-1}) \simeq (\mathsf{M}_{\le b}, \mathsf{M}_{\le a})$ splits into short exact sequences as in (2.2). In particular, we have $H_\ell(\mathsf{M}) \cong H_\ell(\mathsf{M}, \mathsf{M} \cap \mathcal{F}^{\ell-1})$, and M has the homotopy type of a space obtained from the hyperplane section $\mathsf{M} \cap \mathcal{F}^{\ell-1}$ by attaching precisely $b_\ell(\mathsf{M})$ ℓ-cells. $\square$

A similar proof of minimality was recently given by Yoshinaga [Y].

4. Local Systems

As noted in the introduction, the cohomology of the complement of a complex hyperplane arrangement with coefficients in a (complex) local system is of interest in the study of multivariable hypergeometric integrals, among other applications.

Let $\mathcal{A}$ be a hyperplane arrangement in the complex vector space $V \cong \mathbb{C}^\ell$. Let $\rho : \pi_1(\mathsf{M}) \to \mathrm{GL}_m(\mathbb{C})$ be a complex representation of the fundamental group of the complement M of $\mathcal{A}$, and denote by $\mathcal{L}$ the corresponding rank m local system of coefficients on M. For such a local system, stratified Morse theory was used in [C1] to construct a complex $(K^\bullet(\mathcal{A}), \Delta^\bullet)$, the cohomology of which is naturally isomorphic to $H^\bullet(\mathsf{M}; \mathcal{L})$. We recall this construction briefly.

Let $\mathcal{F}$ be a complete flag in V which is transverse to the stratification determined by $\mathcal{A}$ as in (3.1). Let $\mathsf{M}^q = \mathcal{F}^q \cap \mathsf{M}$ for each q. Let $K^q = H^q(\mathsf{M}^q, \mathsf{M}^{q-1}; \mathcal{L})$, and denote by Δ^q the boundary homomorphism $H^q(\mathsf{M}^q, \mathsf{M}^{q-1}; \mathcal{L}) \to H^{q+1}(\mathsf{M}^{q+1}, \mathsf{M}^q; \mathcal{L})$ of the triple $(\mathsf{M}^{q+1}, \mathsf{M}^q, \mathsf{M}^{q-1})$. The following compiles several results from [C1].

Theorem 4.1. *Let $\mathcal{L}$ be the complex local system on M corresponding to the representation $\rho : \pi_1(\mathsf{M}) \to \mathrm{GL}_m(\mathbb{C})$.*

1. *For each q, $0 \le q \le \ell$, we have $H^i(\mathsf{M}^q, \mathsf{M}^{q-1}; \mathcal{L}) = 0$ if $i \ne q$, and $\dim_{\mathbb{C}} H^q(\mathsf{M}^q, \mathsf{M}^{q-1}; \mathcal{L}) = m \cdot b_q(\mathsf{M})$.*
2. *The system of complex vector spaces and linear maps $(K^\bullet, \Delta^\bullet)$,*

$$K^0 \xrightarrow{\Delta^0} K^1 \xrightarrow{\Delta^1} K^2 \longrightarrow \cdots \longrightarrow K^{\ell-1} \xrightarrow{\Delta^{\ell-1}} K^\ell,$$

is a complex ($\Delta^{q+1} \circ \Delta^q = 0$). The cohomology of this complex is naturally isomorphic to $H^\bullet(\mathsf{M}; \mathcal{L})$, the cohomology of M with coefficients in $\mathcal{L}$. $\square$

Corollary 4.2 ([C2]). *For the rank m local system $\mathcal{L}$, let $\beta_q = \dim_{\mathbb{C}} H^q(\mathsf{M};\mathcal{L})$, and write $b_q = b_q(\mathsf{M})$. Then, for $0 \le q \le \ell$, we have*

$$\beta_q \le m \cdot b_q,$$

and

$$\beta_q - \beta_{q-1} + \cdots \pm \beta_0 \le m \cdot (b_q - b_{q-1} + \cdots \pm b_0). \quad \square$$

These are the weak and strong Morse inequalities arising from the complex $(K^\bullet, \Delta^\bullet)$ since $\dim_{\mathbb{C}} K^q = m \cdot b_q$. In particular, for any complex local system, the cohomology groups $H^q(\mathsf{M};\mathcal{L})$ are finite dimensional, resolving a question raised by Aomoto and Kita [AK] in the context of rank one local systems.

Remark 4.3. Let W be the minimal CW-complex resulting from (inductive) application of Theorem 3.2. The complex $(K^\bullet, \Delta^\bullet)$ may be realized as the cellular (co)chain complex of W with coefficients in the local system $\mathcal{L}$.

In the rest of this paper we focus on rank one local systems. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a set of complex weights for the hyperplanes of $\mathcal{A}$. Let $t_j = \exp(-2\pi \mathrm{i}\, \lambda_j)$ and $\mathbf{t} = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n$. Associated to $\boldsymbol{\lambda}$, we have a rank one representation $\rho : \pi_1(\mathsf{M}) \to \mathbb{C}^*$, given by $\rho(\gamma_j) = t_j$, where γ_j is any meridian loop about the hyperplane H_j of $\mathcal{A}$, and a corresponding rank one local system $\mathcal{L} = \mathcal{L}_{\mathbf{t}} = \mathcal{L}_{\boldsymbol{\lambda}}$ on M. Note that weights $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ yield identical representations and local systems if $\boldsymbol{\lambda} - \boldsymbol{\lambda}' \in \mathbb{Z}^n$.

The dimensions of the terms, K^q, of the complex $(K^\bullet, \Delta^\bullet)$ are independent of the local system $\mathcal{L}$. For a rank one local system, they are given by $\dim K^q = b_q(\mathsf{M})$. In this context, write $\Delta^\bullet = \Delta^\bullet(\mathbf{t})$ to indicate the dependence of the complex on $\mathbf{t}$, and view these boundary maps as functions of $\mathbf{t}$. Let $\Lambda = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the ring of complex Laurent polynomials in n commuting variables.

Theorem 4.4 ([CO1]). *For an arrangement $\mathcal{A}$ of n hyperplanes with complement M, there exists a universal complex $(\mathsf{K}^\bullet, \Delta^\bullet(\mathbf{x}))$ with the following properties:*

1. *The terms are free Λ-modules, whose ranks are given by the Betti numbers of M, $\mathsf{K}^q \simeq \Lambda^{b_q(\mathcal{A})}$.*
2. *The boundary maps, $\Delta^q(\mathbf{x}) : \mathsf{K}^q \to \mathsf{K}^{q+1}$ are Λ-linear.*
3. *For each $\mathbf{t} \in (\mathbb{C}^*)^n$, the specialization $\mathbf{x} \mapsto \mathbf{t}$ yields the complex $(K^\bullet, \Delta^\bullet(\mathbf{t}))$, the cohomology of which is isomorphic to $H^\bullet(\mathsf{M};\mathcal{L}_{\mathbf{t}})$, the cohomology of M with coefficients in the local system associated to $\mathbf{t}$.* $\square$

The entries of the boundary maps $\Delta^q(\mathbf{x})$ are elements of the Laurent polynomial ring Λ, the coordinate ring of the complex algebraic n-torus. Via the specialization $\mathbf{x} \mapsto \mathbf{t} \in (\mathbb{C}^*)^n$, we view them as holomorphic functions $(\mathbb{C}^*)^n \to \mathbb{C}$. Similarly, for each q, we view $\Delta^q(\mathbf{x})$ as a holomorphic map $\Delta^q : (\mathbb{C}^*)^n \to \mathrm{Mat}(\mathbb{C})$, $\mathbf{t} \mapsto \Delta^q(\mathbf{t})$ from the complex torus to matrices with complex entries.

Remark 4.5. Let W be the minimal CW-complex resulting from application of Theorem 3.2, and let $\widetilde{W}$ be the universal cover of W. The complex $(\mathsf{K}^\bullet, \Delta^\bullet(\mathbf{x}))$ may be realized as $\mathrm{Hom}^G(C_\bullet(\widetilde{W}), \Lambda)$, where $G = \pi_1(W) = \pi_1(\mathsf{M})$, $C_\bullet(\widetilde{W})$ is the (cellular) chain complex of $\widetilde{W}$, and $\Lambda \cong \mathbb{C}[\mathbb{Z}^n]$ is the G-module corresponding to the action of G on the abelianization $G/[G,G] = H_1(W) = \mathbb{Z}^n$ by (left) translation. In [DP2], Dimca and Papadima show that the complex $C_\bullet(\widetilde{W})$ is itself an invariant of the arrangement $\mathcal{A}$.

The universal complex $\mathsf{K}^\bullet$ is closely related to another universal complex defined by Aomoto [A2] using the Orlik-Solomon algebra $A(\mathcal{A})$. This graded algebra, isomorphic to the cohomology $H^*(\mathsf{M};\mathbb{C})$ (see [OS, OT1]), is the quotient of the exterior algebra $E(\mathcal{A})$ generated by 1-dimensional classes e_j, $1 \le j \le n$, by a homogeneous ideal $I(\mathcal{A})$. Let $[n] = \{1,\ldots,n\}$. Refer to the hyperplanes by their subscripts and order them accordingly. Given $S \subset [n]$, denote the flat $\bigcap_{j\in S} H_j$ by $\cap S$. If $\cap S \neq \emptyset$, call S independent if the codimension of $\cap S$ in V is equal to $|S|$, and dependent if $\operatorname{codim}(\cap S) < |S|$. If $S = (j_1, j_2, \ldots, j_q)$, let $e_S = e_{j_1}e_{j_2}\cdots e_{j_q}$ denote the corresponding basis element of the exterior algebra. Define $\partial e_S = \sum_{p=1}^{q}(-1)^{p-1}e_{S\setminus\{j_p\}}$. The ideal $I(\mathcal{A})$ is generated by

$$\{\partial e_S \mid S \text{ is dependent}\} \bigcup \{e_S \mid \cap S = \emptyset\}.$$

For $S \subset [n]$, let a_S denote the image of e_S in $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$. The algebra $A(\mathcal{A})$ has a $\mathbb{C}$-basis called the **nbc** basis. A subset S of $[n]$ is a *circuit* if it is a minimally dependent set: S is dependent but every nontrivial subset of S is independent. Call $T = (j_1 < \cdots < j_p) \subset [n]$ a broken circuit if there exists $k \in [n]$ so that $k < j_1$ and (k, T) is a circuit. The **nbc** basis consists of all elements a_S of $A(\mathcal{A})$ corresponding to subsets S of $[n]$ which contain no broken circuit [OT1].

Let $a_{\boldsymbol{\lambda}} = \sum_{j=1}^{n} \lambda_j a_j \in A^1(\mathcal{A})$ and note that $a_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}} = 0$ because $A(\mathcal{A})$ is a quotient of an exterior algebra. Thus we have a complex $(A^\bullet(\mathcal{A}), a_{\boldsymbol{\lambda}})$. Let $\boldsymbol{y} = \{y_1,\ldots,y_n\}$ be a set of indeterminates in one-to-one correspondence with the hyperplanes of $\mathcal{A}$. Let $R = \mathbb{C}[\boldsymbol{y}]$ be the polynomial ring in $\boldsymbol{y}$. Define a graded R-algebra: $\mathsf{A}^\bullet = \mathsf{A}^\bullet(\mathcal{A}) = R \otimes_{\mathbb{C}} A^\bullet(\mathcal{A})$. Let $a_{\boldsymbol{y}} = \sum_{j=1}^{n} y_j \otimes a_j \in \mathsf{A}^1$. The complex $(\mathsf{A}^\bullet(\mathcal{A}), a_{\boldsymbol{y}})$

$$0 \to \mathsf{A}^0(\mathcal{A}) \xrightarrow{a_{\boldsymbol{y}}} \mathsf{A}^1(\mathcal{A}) \xrightarrow{a_{\boldsymbol{y}}} \ldots \xrightarrow{a_{\boldsymbol{y}}} \mathsf{A}^\ell(\mathcal{A}) \to 0 \tag{4.1}$$

is called the *Aomoto complex*. Its specialization $\mathbf{y} \mapsto \boldsymbol{\lambda}$ is the complex $(A^\bullet(\mathcal{A}), a_{\boldsymbol{\lambda}})$.

Theorem 4.6 ([CO1]). *For any arrangement $\mathcal{A}$, the Aomoto complex $(\mathsf{A}^\bullet, a_{\mathbf{y}})$ is chain equivalent to the linearization of the universal complex $(\mathsf{K}^\bullet, \Delta^\bullet(\mathbf{x}))$ at the point $\mathbf{1} = (1,\ldots,1) \in (\mathbb{C}^*)^n$.* □

For certain classes of arrangements, the boundary maps of the universal complex $(\mathsf{K}^\bullet, \Delta^\bullet(\mathbf{x}))$ may be described explicitly. See, for instance, Hattori [H] for general position arrangements. In the case where the arrangement is defined by real equations, progress on this problem has been recently made by Yoshinaga [Y]. However, for an arbitrary arrangement, these boundary maps are not known. Consequently, while the complex $(K^\bullet, \Delta^\bullet(\mathbf{t}))$ computes local system cohomology in principle, we do not know how to calculate the groups $H^q(\mathsf{M};\mathcal{L}_{\mathbf{t}})$ explicitly for arbitrary weights.

It is an interesting question to determine the stratification of the space of all weights with respect to the local system cohomology groups. Each point $\mathbf{t} \in (\mathbb{C}^*)^n$ gives rise to a local system $\mathcal{L} = \mathcal{L}_{\mathbf{t}}$ on the complement M. Define the *characteristic varieties*

$$\Sigma^q_m(\mathsf{M}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \dim H^q(\mathsf{M};\mathcal{L}_{\mathbf{t}}) \ge m\}.$$

These loci are algebraic subvarieties of $(\mathbb{C}^*)^n$, which are invariants of the homotopy type of M. See Arapura [Ar] and Libgober [L1] for detailed discussions of these varieties in the contexts of quasiprojective varieties and plane algebraic curves. The characteristic varieties are closely related to the resonance varieties.

Each point $\boldsymbol{\lambda} \in \mathbb{C}^n$ gives rise to an element $a_{\boldsymbol{\lambda}} \in A^1$ of the Orlik-Solomon algebra $A^\bullet = A^\bullet(\mathcal{A})$. Define the *resonance varieties*

$$\mathcal{R}^q_m(A) = \{\lambda \in \mathbb{C}^n \mid \dim H^q(A^\bullet, a_{\boldsymbol{\lambda}}) \geq m\}.$$

These subvarieties of $\mathbb{C}^n$ are invariants of the Orlik-Solomon algebra $A(\mathcal{A})$. See Falk [F] and Libgober and Yuzvinsky [LY] for detailed discussions of these varieties.

Theorem 4.7 ([CO1])**.** *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ with complement M and Orlik-Solomon algebra $A^\bullet$. For each q and m, the resonance variety $\mathcal{R}^q_m(A)$ coincides with the tangent cone of the characteristic variety $\Sigma^q_m(\mathsf{M})$ at the point $\mathbf{1} = (1, \ldots, 1) \in (\mathbb{C}^*)^n$.* □

The characteristic varieties are known to be unions of torsion-translated subtori of $(\mathbb{C}^*)^n$, see [Ar]. In particular, all irreducible components of $\Sigma^q_m(\mathsf{M})$ passing through $\mathbf{1}$ are subtori of $(\mathbb{C}^*)^n$. Consequently, all irreducible components of the tangent cone are linear subspaces of $\mathbb{C}^n$.

Corollary 4.8. *For each q and m, the resonance variety $\mathcal{R}^q_m(A)$ is the union of an arrangement of subspaces in $\mathbb{C}^n$.* □

For $q = 1$, these results were established by Cohen and Suciu [CS], see also Libgober and Yuzvinsky [L1, LY]. For the discriminantal arrangements of Schechtman and Varchenko [SV2], they were established in [C3]. In particular, as conjectured by Falk [F, Conjecture 4.7], the resonance varieties $\mathcal{R}^q_m(A)$ were known to be unions of linear subspaces in these instances. Corollary 4.8 above resolves this conjecture positively for all arrangements in all dimensions. Theorem 4.7 and Corollary 4.8 have been obtained by Libgober in a more general situation, see [L2].

There are examples of arrangements for which the characteristic varieties contain (positive dimensional) components which do not pass through $\mathbf{1}$ and hence cannot be detected by the resonance variety, see Suciu [Su]. In some of these cases, the local system cohomology is nontrivial, while the cohomology of the Orlik-Solomon complex vanishes.

5. Moduli Spaces

In the rest of the paper we pass from consideration of a fixed arrangement to the study of all arrangements of a given combinatorial type. Fix a pair (ℓ, n) with $n \geq \ell \geq 1$ and consider families of essential ℓ-arrangements with n linearly ordered hyperplanes. In order to define the notions of combinatorial type and degeneration, we must allow for the coincidence of several hyperplanes. We call these new objects *multi-arrangements*. If there is no coincidence, we call the arrangement *simple*.

Introduce coordinates $u_1, \ldots, u_\ell$ in V and choose a degree one polynomial $\alpha_j = b_{j,0} + \sum_{k=1}^{\ell} b_{j,k} u_k$ for the hyperplane $H_j \in \mathcal{A}$ so H_j is defined by $\alpha_j = 0$. Note that α_j is unique up to a constant. Embed V in projective space $\mathbb{CP}^\ell$ and call the complement of V the infinite hyperplane, H_{n+1}, defined by $u_0 = 0$. We call $\mathcal{A}_\infty = \mathcal{A} \bigcup H_{n+1}$ the *projective closure* of $\mathcal{A}$. It is an arrangement in $\mathbb{CP}^\ell$. Give H_{n+1} the weight $\lambda_{n+1} = -\sum_{j=1}^{n} \lambda_j$. We agree that the hyperplane at infinity, H_{n+1}, is largest in the ordering. We may therefore view the projective closure of

the arrangement as an $(n+1)\times(\ell+1)$ matrix of complex numbers

$$\mathsf{b} = \begin{pmatrix} b_{1,0} & b_{1,1} & \cdots & b_{1,\ell} \\ b_{2,0} & b_{2,1} & \cdots & b_{2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & \cdots & b_{n,\ell} \\ 1 & 0 & \cdots & 0 \end{pmatrix} \tag{5.1}$$

whose rows correspond to the hyperplanes of $\mathcal{A}_\infty$. Thus $(\mathbb{CP}^\ell)^n$ may be viewed as the moduli space of all ordered multi-arrangements in $\mathbb{CP}^\ell$ with n hyperplanes together with the hyperplane at infinity.

Given an arrangement $\mathcal{A}$, the set $S = (j_1, \dots, j_q)$ is dependent (in the projective closure) if the corresponding row vectors of (5.1) are linearly dependent. Let $\mathrm{Dep}(\mathcal{A})_q$ be the set of dependent q-tuples and let $\mathrm{Dep}(\mathcal{A}) = \bigcup_{q>1} \mathrm{Dep}(\mathcal{A})_q$. Two essential simple arrangements are combinatorially equivalent if and only if they have the same dependent sets. We call $\mathcal{T}$ their *combinatorial type* and write $\mathrm{Dep}(\mathcal{T})$. Note that an arbitrary collection of subsets of $[n+1]$ is not necessarily realizable as a dependent (or independent) set. For example, the collection $\{123, 124, 134\}$ is not realizable as a dependent set, since these dependencies imply the dependence of 234.

The combinatorial type is, in fact, determined by $\mathrm{Dep}(\mathcal{T})_{\ell+1}$, see, for instance, Terao [T]. Given a subset $J \subset [n+1]$ of cardinality $\ell+1$, let $\Delta_J(\mathsf{b})$ denote the determinant of the $(\ell+1)\times(\ell+1)$ submatrix of b whose rows are specified by J. Given a realizable type $\mathcal{T}$, the moduli space of type $\mathcal{T}$ is

$$\mathsf{X}(\mathcal{T}) = \{\mathsf{b} \in (\mathbb{CP}^\ell)^n \mid \Delta_J(\mathsf{b}) = 0 \text{ for } J \in \mathrm{Dep}(\mathcal{T})_{\ell+1},\ \Delta_J(\mathsf{b}) \neq 0 \text{ else}\}.$$

If $\mathcal{G}$ is the type of a general position arrangement, then $\mathrm{Dep}(\mathcal{G}) = \emptyset$ and the moduli space $\mathsf{X}(\mathcal{G})$ is a dense, open subset of $(\mathbb{CP}^\ell)^n$. Define a partial order on combinatorial types as follows: $\mathcal{T} \geq \mathcal{T}' \iff \mathrm{Dep}(\mathcal{T}) \subseteq \mathrm{Dep}(\mathcal{T}')$. The combinatorial type $\mathcal{G}$ is the maximal element with respect to this partial order. Write $\mathcal{T} > \mathcal{T}'$ if $\mathrm{Dep}(\mathcal{T}) \subsetneq \mathrm{Dep}(\mathcal{T}')$. If $\mathcal{T} > \mathcal{T}'$, we say that $\mathcal{T}$ *covers* $\mathcal{T}'$ and $\mathcal{T}'$ is a *degeneration* of $\mathcal{T}$ if there is no realizable combinatorial type $\mathcal{T}''$ with $\mathcal{T} > \mathcal{T}'' > \mathcal{T}'$. In this case we define the relative dependence set

$$\mathrm{Dep}(\mathcal{T}', \mathcal{T}) = \mathrm{Dep}(\mathcal{T}') \setminus \mathrm{Dep}(\mathcal{T}).$$

Let

$$\mathsf{Y}(\mathcal{T}) = \{\mathsf{b} \in (\mathbb{CP}^\ell)^n \mid \Delta_J(\mathsf{b}) \neq 0 \text{ for } J \notin \mathrm{Dep}(\mathcal{T})_{\ell+1}\}.$$

Then the moduli space of type $\mathcal{T}$ may be realized as

$$\mathsf{X}(\mathcal{T}) = \{\mathsf{b} \in \mathsf{Y}(\mathcal{T}) \mid \Delta_J(\mathsf{b}) = 0 \text{ for } J \in \mathrm{Dep}(\mathcal{T})_{\ell+1}\}.$$

Note that $\mathsf{X}(\mathcal{G}) = \mathsf{Y}(\mathcal{G})$. For any other type $\mathcal{T}$, the moduli space $\mathsf{X}(\mathcal{G})$ may be realized as

$$\mathsf{X}(\mathcal{G}) = \{\mathsf{b} \in \mathsf{Y}(\mathcal{T}) \mid \Delta_J(\mathsf{b}) \neq 0 \text{ for } J \in \mathrm{Dep}(\mathcal{T})_{\ell+1}\}.$$

If $\mathcal{T} \neq \mathcal{G}$, then $\mathsf{X}(\mathcal{T})$ and $\mathsf{X}(\mathcal{G})$ are disjoint subspaces of $\mathsf{Y}(\mathcal{T})$. Let $i_\mathcal{T} : \mathsf{X}(\mathcal{G}) \to \mathsf{Y}(\mathcal{T})$ and $j_\mathcal{T} : \mathsf{X}(\mathcal{T}) \to \mathsf{Y}(\mathcal{T})$ denote the natural inclusions. We showed in [CO3] that for any combinatorial type $\mathcal{T}$, the inclusion $i_\mathcal{T} : \mathsf{X}(\mathcal{G}) \to \mathsf{Y}(\mathcal{T})$ induces a surjection $(i_\mathcal{T})_* : H_1(\mathsf{X}(\mathcal{G})) \to H_1(\mathsf{Y}(\mathcal{T}))$.

For the type $\mathcal{G}$ of general position arrangements, the closure of the moduli space is $\overline{\mathsf{X}}(\mathcal{G}) = (\mathbb{CP}^\ell)^n$. The divisor $\mathsf{D}(\mathcal{G}) = \overline{\mathsf{X}}(\mathcal{G}) \setminus \mathsf{X}(\mathcal{G})$ is given by $\mathsf{D}(\mathcal{G}) = \bigcup_J \mathsf{D}_J$,

whose components, $\mathsf{D}_J = \{\mathsf{b} \in (\mathbb{CP}^\ell)^n \mid \Delta_J(\mathsf{b}) = 0\}$, are irreducible hypersurfaces indexed by $J = \{j_1, \dots, j_{\ell+1}\}$. Choose a basepoint $\mathbf{c} \in \mathsf{X}(\mathcal{G})$, and for each $\ell+1$ element subset J of $[n+1]$, let $\mathbf{d}_J$ be a generic point in D_J. Let Γ_J be a meridian loop based at $\mathbf{c}$ in $\mathsf{X}(\mathcal{G})$ about the point $\mathbf{d}_J \in \mathsf{D}_J$. Note that $\mathbf{c} \in \mathsf{Y}(\mathcal{T})$ and that Γ_J is a (possibly null-homotopic) loop in $\mathsf{Y}(\mathcal{T})$ for any combinatorial type $\mathcal{T}$. We showed in [CO3] that for any combinatorial type $\mathcal{T}$, the homology group $H_1(\mathsf{Y}(\mathcal{T}))$ is generated by the classes $\{[\Gamma_J] \mid J \notin \mathrm{Dep}(\mathcal{T})_{\ell+1}\}$. In particular, the homology group $H_1(\mathsf{X}(\mathcal{G}))$ is generated by the classes $[\Gamma_J]$, where J ranges over all $\ell+1$ element subsets of $[n+1]$.

It is easy to see that the moduli space $\mathsf{X}(\mathcal{T}')$ has complex codimension one in the closure $\overline{\mathsf{X}}(\mathcal{T})$ if and only if $\mathcal{T}$ covers $\mathcal{T}'$. The next theorem is essential for later results.

Theorem 5.1 ([CO3])**.** *Let $\mathcal{T}$ be a combinatorial type which covers the type $\mathcal{T}'$. Let b' be a point in $\mathsf{X}(\mathcal{T}')$, and $\gamma \in \pi_1(\mathsf{X}(\mathcal{T}), \mathsf{b})$ a simple loop in $\mathsf{X}(\mathcal{T})$ about b'. Then the homology class $[\gamma]$ satisfies*

$$(j_{\mathcal{T}})_*([\gamma]) = \sum_{J \in \mathrm{Dep}(\mathcal{T}', \mathcal{T})} m_J \cdot [\Gamma_J], \tag{5.2}$$

where m_J is the order of vanishing of the restriction of Δ_J to $\overline{\mathsf{X}}(\mathcal{T})$ along $\mathsf{X}(\mathcal{T}')$. □

6. Gauss-Manin Connections

The moduli space $\mathsf{X}(\mathcal{T})$ is not necessarily connected. The existence of a combinatorial type whose moduli space has at least two components follows from examples of Rybnikov [Ry]. Let $\mathsf{B}(\mathcal{T})$ be a smooth component of the moduli space. Corresponding to each $\mathsf{b} \in \mathsf{B}(\mathcal{T})$, we have an arrangement $\mathcal{A}_\mathsf{b}$, combinatorially equivalent to $\mathcal{A}$, with hyperplanes defined by the first n rows of the matrix equation $\mathsf{b} \cdot \tilde{\mathsf{u}} = 0$, where $\tilde{\mathsf{u}} = \begin{pmatrix} 1 & u_1 & \cdots & u_\ell \end{pmatrix}^\top$. Let $\mathsf{M}_\mathsf{b} = M(\mathcal{A}_\mathsf{b})$ be the complement of $\mathcal{A}_\mathsf{b}$. Let

$$\mathsf{M}(\mathcal{T}) = \{(\mathsf{b}, \mathsf{u}) \in (\mathbb{CP}^\ell)^n \times \mathbb{C}^\ell \mid \mathsf{b} \in \mathsf{B}(\mathcal{T}) \text{ and } \mathsf{u} \in \mathsf{M}_\mathsf{b}\},$$

and define $\pi_\mathcal{T} : \mathsf{M}(\mathcal{T}) \to \mathsf{B}(\mathcal{T})$ by $\pi_\mathcal{T}(\mathsf{b}, \mathsf{u}) = \mathsf{b}$. Since $\mathsf{B}(\mathcal{T})$ is connected by assumption, a result of Randell [Ra1] implies that $\pi_\mathcal{T} : \mathsf{M}(\mathcal{T}) \to \mathsf{B}(\mathcal{T})$ is a bundle, with fiber $\pi_\mathcal{T}^{-1}(\mathsf{b}) = \mathsf{M}_\mathsf{b}$.

For each $\mathsf{b} \in \mathsf{B}(\mathcal{T})$, weights $\boldsymbol{\lambda}$ define a local system $\mathcal{L}_\mathsf{b}$ on M_b. Since $\pi_\mathcal{T} : \mathsf{M}(\mathcal{T}) \to \mathsf{B}(\mathcal{T})$ is locally trivial, there is an associated flat vector bundle $\pi^q : \mathbf{H}^q(\mathcal{L}) \to \mathsf{B}(\mathcal{T})$, with fiber $(\pi^q)^{-1}_\mathcal{L}(\mathsf{b}) = H^q(\mathsf{M}_\mathsf{b}; \mathcal{L}_\mathsf{b})$ at $\mathsf{b} \in \mathsf{B}(\mathcal{T})$ for each q, $0 \le q \le \ell$. Fixing a basepoint $\mathsf{b} \in \mathsf{B}(\mathcal{T})$, the operation of parallel translation of fibers over curves in $\mathsf{B}(\mathcal{T})$ provides a complex representation

$$\Psi^q_\mathcal{T} : \pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b}) \longrightarrow \mathrm{Aut}_\mathbb{C}(H^q(\mathsf{M}_\mathsf{b}; \mathcal{L}_\mathsf{b})). \tag{6.1}$$

The cohomology of the Morse theoretic complex $K^\bullet(\mathcal{A}_\mathsf{b})$ is isomorphic to the cohomology of M_b with coefficients in the local system $\mathcal{L}_\mathsf{b}$. The fundmental group of $\mathsf{B}(\mathcal{T})$ acts by chain automorphisms on this complex, see [CO2, Cor. 3.2], yielding a representation

$$\psi^\bullet_\mathcal{T} : \pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b}) \longrightarrow \mathrm{Aut}_\mathbb{C}(K^\bullet(\mathcal{A}_\mathsf{b})).$$

Theorem 6.1. *The representation $\Psi^q_\mathcal{T} : \pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b}) \to \mathrm{Aut}_\mathbb{C}(H^q(\mathsf{M}_\mathsf{b}; \mathcal{L}_\mathsf{b}))$ is induced by the representation $\psi^\bullet_\mathcal{T} : \pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b}) \to \mathrm{Aut}_\mathbb{C}(K^\bullet(\mathcal{A}_\mathsf{b}))$.* □

The vector bundle $\pi^q : \mathbf{H}^q(\mathcal{L}) \to \mathsf{B}(\mathcal{T})$ supports a Gauss-Manin connection corresponding to the representation (6.1). Over a manifold X, there is a well known equivalence between complex local systems and complex vector bundles equipped with flat connections, see [D, Ko]. Let $\mathbf{V} \to X$ be such a bundle, with connection ∇. The latter is a $\mathbb{C}$-linear map $\nabla : \mathcal{E}^0(\mathbf{V}) \to \mathcal{E}^1(\mathbf{V})$, where $\mathcal{E}^p(\mathbf{V})$ denotes the complex p-forms on X with values in $\mathbf{V}$, which satisfies $\nabla(f\sigma) = \sigma df + f\nabla(\sigma)$ for a function f and $\sigma \in \mathcal{E}^0(\mathbf{V})$. The connection extends to a map $\nabla : \mathcal{E}^p(\mathbf{V}) \to \mathcal{E}^{p+1}(\mathbf{V})$ for $p \geq 0$, and is flat if the curvature $\nabla \circ \nabla$ vanishes. Call two connections ∇ and ∇' on $\mathbf{V}$ isomorphic if ∇' is obtained from ∇ by a gauge transformation, $\nabla' = g \circ \nabla \circ g^{-1}$ for some $g : X \to \mathrm{Hom}(\mathbf{V}, \mathbf{V})$.

The aforementioned equivalence is given by $(\mathbf{V}, \nabla) \mapsto \mathbf{V}^\nabla$, where $\mathbf{V}^\nabla$ is the local system, or locally constant sheaf, of horizontal sections $\{\sigma \in \mathcal{E}^0(\mathbf{V}) \mid \nabla(\sigma) = 0\}$. There is also a well known equivalence between local systems on X and finite dimensional representations of the fundamental group of X. Note that isomorphic connections give rise to the same representation. Under these equivalences, the local system on $X = \mathsf{B}(\mathcal{T})$ induced by the representation $\Psi^q_{\mathcal{T}}$ corresponds to a flat connection on the vector bundle $\pi^q : \mathbf{H}^q(\mathcal{L}) \to \mathsf{B}(\mathcal{T})$, the Gauss-Manin connection.

Let $\gamma \in \pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b})$, and let $g : \mathbb{S}^1 \to \mathsf{B}(\mathcal{T})$ be a representative loop. Pulling back the bundle $\pi^q : \mathbf{H}^q(\mathcal{L}) \to \mathsf{B}(\mathcal{T})$ and the Gauss-Manin connection ∇, we obtain a flat connection $g^*(\nabla)$ on the vector bundle over the circle corresponding to the representation of $\pi_1(\mathbb{S}^1, 1) = \langle \zeta \rangle = \mathbb{Z}$ given by $\zeta \mapsto \Psi^q_{\mathcal{T}}(\gamma)$. This vector bundle is trivial since any map from the circle to the relevant classifying space is null-homotopic. Specifying the flat connection $g^*(\nabla)$ amounts to choosing a logarithm of $\Psi^q_{\mathcal{T}}(\gamma)$. The connection $g^*(\nabla)$ is determined by a connection 1-form $dz/z \otimes \Omega^q_{\mathcal{T}}(\gamma)$, where the connection matrix $\Omega^q_{\mathcal{T}}(\gamma)$ corresponding to γ satisfies $\Psi^q_{\mathcal{T}}(\gamma) = \exp(-2\pi\,\mathrm{i}\,\Omega^q_{\mathcal{T}}(\gamma))$. If γ and $\hat{\gamma}$ are conjugate in $\pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b})$, then the resulting connection matrices are conjugate, and the corresponding connections on the trivial vector bundle over the circle are isomorphic. In this sense, the connection matrix $\Omega^q_{\mathcal{T}}(\gamma)$ is determined by the homology class $[\gamma]$ of γ.

In the special case when $\mathcal{T}$ covers $\mathcal{T}'$ and $\gamma \in \pi_1(\mathsf{B}(\mathcal{T}), \mathsf{b})$ is a simple loop linking $\mathsf{B}(\mathcal{T}')$ in $\overline{\mathsf{B}(\mathcal{T})}$, we denote the corresponding Gauss-Manin connection matrix in the bundle $\pi^q : \mathbf{H}^q(\mathcal{L}) \to \mathsf{B}(\mathcal{T})$ by $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$. The relationship between the homology classes of the loop γ and the loops Γ_J in the moduli space of a general position arrangement exhibited in Theorem 5.1 suggests an analogous relationship between the corresponding Gauss-Manin endomorphisms.

For nonresonant weights $\boldsymbol{\lambda}$, this relationship is pursued in [CO3]. In this situation, the local system cohomology is concentrated in the top dimension, and is isomorphic to the cohomology of the Orlik-Solomon complex, $H^\ell(\mathsf{M}; \mathcal{L}_{\boldsymbol{\lambda}}) \cong H^\ell(A^\bullet(\mathcal{A}), a_{\boldsymbol{\lambda}})$. Moreover, there is a surjection $P : H^\ell(A(\mathcal{G}), e_{\boldsymbol{\lambda}}) \twoheadrightarrow H^\ell(A^\bullet(\mathcal{A}), a_{\boldsymbol{\lambda}})$ from the cohomology of the Orlik-Solomon complex of a general position arrangement to that of $\mathcal{A}$, see [CO3, Theorem 6.5].

Theorem 6.2. *Let $\mathcal{T}$ be a combinatorial type which covers the type $\mathcal{T}'$. Let $\boldsymbol{\lambda}$ be a collection of weights which are nonresonant for type $\mathcal{T}$ (and hence for type $\mathcal{G}$). Then the Gauss-Manin endomorphism $\Omega^\ell_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$ is determined by the equation*

$$P \cdot \Omega^\ell_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T})) = \Big(\sum_{J \in \mathrm{Dep}_{\ell+1}(\mathcal{T}', \mathcal{T})} m_J \cdot \Omega^\ell_{\mathcal{L}}(\mathsf{B}(\mathcal{T}_J), \mathsf{B}(\mathcal{G})) \Big) \cdot P,$$

where $\mathcal{T}_J$ is the combinatorial type of an arrangement for which J is the only dependent set of size $\ell+1$, and $\Omega^{\ell}_{\mathcal{L}}(\mathsf{B}(\mathcal{T}_J), \mathsf{B}(\mathcal{G})) \in \operatorname{End} H^{\ell}(A(\mathcal{G}), e_{\boldsymbol{\lambda}})$ is the corresponding Aomoto-Kita Gauss-Manin connection matrix. $\square$

For arbitrary weights $\boldsymbol{\lambda}$, Theorems 5.1 and 6.2 motivated the construction of formal connections in the Aomoto complex of a general position arrangement in [CO4]. These are discussed in Section 7.

The Gauss-Manin connection in local system cohomology has combinatorial analogs. We have the vector bundle $\mathbf{A}^q \to \mathsf{B}(\mathcal{T})$, whose fiber at b is $A^q(\mathcal{A}_{\mathsf{b}})$, the q-th graded component of the Orlik-Solomon algebra of the arrangement $\mathcal{A}_{\mathsf{b}}$. The **nbc** basis provides a global trivialization of this bundle. Given weights $\boldsymbol{\lambda}$, the cohomology of the complex $(A^{\bullet}(\mathcal{A}_{\mathsf{b}}), a_{\boldsymbol{\lambda}})$ gives rise to the flat vector bundle $\mathbf{H}^q(A) \to \mathsf{B}(\mathcal{T})$ whose fiber at b is the q-th cohomology group of the Orlik-Solomon algebra, $H^q(A^{\bullet}(\mathcal{A}_{\mathsf{b}}), a_{\boldsymbol{\lambda}})$. Like their topological counterparts, these algebraic vector bundles admit flat connections. If $\mathcal{T}$ covers $\mathcal{T}'$, denote the corresponding connection matrix in this cohomology bundle by $\Omega^q_A(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$.

7. Formal Connections

To determine the endomorphisms $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$ and $\Omega^q_A(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$, we define formal connections in the Aomoto complex, $(\mathsf{A}^{\bullet}(\mathcal{G}), a_{\mathbf{y}}))$, of the general position arrangement of n ordered hyperplanes in $\mathbb{C}^{\ell}$. We embed the arrangement in projective space as described above and call the resulting type $\mathcal{G}_{\infty}$. The symmetric group Σ_{n+1} on $n+1$ letters acts on $\mathsf{A}^{\bullet}(\mathcal{G})$, the rank ℓ truncation of the exterior algebra in n variables, by permuting the hyperplanes of $\mathcal{G}_{\infty}$, and on R by permuting the variables y_j, where $y_{n+1} = -\sum_{j=1}^{n} y_j$. In the basis $\{e_j \mid 1 \le j \le n\}$ for the exterior algebra, the action of $\sigma \in \Sigma_{n+1}$ is given by $\sigma(e_i) = e_{\sigma(i)}$ if $\sigma(n+1) = n+1$, and by

$$\sigma(e_i) = \begin{cases} -e_{\sigma(n+1)} & \text{if } \sigma(i) = n+1, \\ e_{\sigma(i)} - e_{\sigma(n+1)} & \text{if } \sigma(i) \neq n+1, \end{cases}$$

if $\sigma(n+1) \neq n+1$. Denote the induced action on the Aomoto complex by $\phi_{\sigma} : \mathsf{A}^{\bullet}(\mathcal{G}) \to \mathsf{A}^{\bullet}(\mathcal{G})$,

$$\phi_{\sigma}(e_{i_1} \cdots e_{i_p} \otimes f(y_1, \ldots, y_n)) = \sigma(e_{i_1}) \cdots \sigma(e_{i_p}) \otimes f(y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

Lemma 7.1. *For each $\sigma \in \Sigma_{n+1}$, the map ϕ_{σ} is a cochain automorphism of the Aomoto complex $(\mathsf{A}^{\bullet}(\mathcal{G}), e_{\mathbf{y}})$.* $\square$

If $T = (i_1, \ldots, i_p) \subset [n]$ is a p-tuple, then we write $e_T = e_{i_1} \cdots e_{i_p}$. Recall that $\partial e_T = \sum_{j=1}^{p} (-1)^{j-1} e_{T \setminus \{i_j\}}$. For $j \in [n]$, let $(j, T) = (j, i_1, \ldots, i_p)$ be the $(p+1)$-tuple which adds j to T as its first entry. For $S = (s_1, \ldots, s_k) \subset [n+1]$, let σ_S denote the permutation $\left(\begin{smallmatrix} 1 & 2 & \cdots & k \\ s_1 & s_2 & \cdots & s_k \end{smallmatrix}\right)$. Write $S \equiv T$ if S and T are equal sets.

Definition 7.2. *Let $T \subset [n]$ be a p-tuple, $S \subset [n+1]$ have size $q+1$, and $j \in [n]$. If $S = S_0 = [q+1]$, define the endomorphism $\tilde{\omega}^{\bullet}_{S_0} : (\mathsf{A}^{\bullet}(\mathcal{G}), e_{\mathbf{y}}) \to (\mathsf{A}^{\bullet}(\mathcal{G}), e_{\mathbf{y}})$ by*

$$\tilde{\omega}^{p}_{S_0}(e_T) = \begin{cases} y_j \partial e_{(j,T)} & \text{if } p = q \text{ and } S_0 \equiv (j, T), \\ e_{\mathbf{y}} \partial e_T & \text{if } p = q+1 \text{ and } S_0 \equiv T, \\ 0 & \text{otherwise.} \end{cases}$$

If $S \neq S_0$, define $\tilde{\omega}^{\bullet}_S = \phi_{\sigma_S} \circ \tilde{\omega}^{\bullet}_{S_0} \circ \phi^{-1}_{\sigma_S}$.

Proposition 7.3 ([CO4]). *For every subset S of $[n+1]$, the map $\tilde{\omega}_S^\bullet$ is a cochain homomorphism of the Aomoto complex $(\mathsf{A}^\bullet(\mathcal{G}), e_{\mathbf{y}}))$.* □

The formal connection endomorphisms $\tilde{\omega}_S^\bullet$ are defined for the Aomoto complex of the general position type $\mathcal{G}$. Our aim is to show that certain linear combinations of these induce endomorphisms of the Aomoto complex of type $\mathcal{T}$ for all pairs of types $\mathcal{T}'$, $\mathcal{T}$ where $\mathcal{T}$ covers $\mathcal{T}'$. This involves multiplicities. Given $S \subset [n+1]$, let $N_S(\mathcal{T}) = N_S(\mathsf{b})$ denote the submatrix of (5.1) with rows specified by S. Let $\operatorname{rank} N_S(\mathcal{T})$ be the size of the largest minor with nonzero determinant. Define the *multiplicity* of S in $\mathcal{T}$ by

$$m_S(\mathcal{T}) = |S| - \operatorname{rank} N_S(\mathcal{T}).$$

It is not hard to see that this definition of multiplicity agrees with the analytic definition in Theorem 5.1. Let

$$\tilde{\omega}(\mathcal{T}', \mathcal{T}) = \sum_{S \in \operatorname{Dep}(\mathcal{T}', \mathcal{T})} m_S(\mathcal{T}') \cdot \tilde{\omega}_S.$$

For an arrangement $\mathcal{A}$, the Orlik-Solomon algebra depends only on the combinatorial type $\mathcal{T}$, so we write $A(\mathcal{A}) = A(\mathcal{T})$. If $\mathcal{A} \subset V \cong \mathbb{C}^\ell$, then $A^q(\mathcal{T}) = 0$ for $q > \ell$. It follows that $A(\mathcal{T})$ may be realized as a quotient of the rank ℓ truncation of the exterior algebra $E(\mathcal{A})$, which is itself the Orlik-Solomon algebra $A(\mathcal{G})$ of the combinatorial type of a general position arrangement. Denote the rank ℓ truncation of the Orlik-Solomon ideal $I(\mathcal{A})$ by $I(\mathcal{T}) = I(\mathcal{A}) \cap A(\mathcal{G})$. Thus $A(\mathcal{T}) = A(\mathcal{G})/I(\mathcal{T})$. The ideal $I(\mathcal{T})$ gives rise to a subcomplex $\mathsf{I}^\bullet(\mathcal{T})$ of the Aomoto complex $\mathsf{A}^\bullet(\mathcal{G})$, and we have an exact sequence of cochain complexes

$$0 \to \mathsf{I}^\bullet(\mathcal{T}) \to \mathsf{A}^\bullet(\mathcal{G}) \to \mathsf{A}^\bullet(\mathcal{T}) \to 0.$$

Theorem 7.4 ([CO4]). *If $\mathcal{T}$ covers $\mathcal{T}'$, then $\tilde{\omega}(\mathcal{T}', \mathcal{T})(\mathsf{I}^\bullet(\mathcal{T})) \subset \mathsf{I}^\bullet(\mathcal{T})$ so there is a commutative diagram*

$$\begin{array}{ccccc}
(\mathsf{I}^\bullet(\mathcal{T}), e_{\mathbf{y}}) & \xrightarrow{\iota} & (\mathsf{A}^\bullet(\mathcal{G}), e_{\mathbf{y}}) & \xrightarrow{p} & (\mathsf{A}^\bullet(\mathcal{T}), a_{\mathbf{y}}) \\
\Big\downarrow{\scriptstyle \tilde{\omega}(\mathcal{T}', \mathcal{T})|_{\mathsf{I}^\bullet(\mathcal{T})}} & & \Big\downarrow{\scriptstyle \tilde{\omega}(\mathcal{T}', \mathcal{T})} & & \Big\downarrow{\scriptstyle \omega(\mathcal{T}', \mathcal{T})} \\
(\mathsf{I}^\bullet(\mathcal{T}), e_{\mathbf{y}}) & \xrightarrow{\iota} & (\mathsf{A}^\bullet(\mathcal{G}), e_{\mathbf{y}}) & \xrightarrow{p} & (\mathsf{A}^\bullet(\mathcal{T}), a_{\mathbf{y}})
\end{array}$$

where $\iota : \mathsf{I}^\bullet(\mathcal{T}) \to \mathsf{A}^\bullet(\mathcal{G})$ is the inclusion, $p : \mathsf{A}^\bullet(\mathcal{G}) \to \mathsf{A}^\bullet(\mathcal{T}) = \mathsf{A}^\bullet(\mathcal{G})/\mathsf{I}^\bullet(\mathcal{T})$ is the natural projection, and $\omega(\mathcal{T}', \mathcal{T}) : \mathsf{A}^\bullet(\mathcal{T}) \to \mathsf{A}^\bullet(\mathcal{T})$ is the induced map. □

We call the map $\omega(\mathcal{T}', \mathcal{T})$ the universal Gauss-Manin endomorphism.

It follows that for given weights $\boldsymbol{\lambda}$, the specialization $\mathbf{y} \mapsto \boldsymbol{\lambda}$ in the chain endomorphism $\omega(\mathcal{T}', \mathcal{T})$ defines a chain endomorphism $\omega_{\boldsymbol{\lambda}}^\bullet(\mathcal{T}', \mathcal{T}) : A^\bullet(\mathcal{T}) \to A^\bullet(\mathcal{T})$. Let $\kappa^q = \ker[a_{\boldsymbol{\lambda}} : A^q(\mathcal{T}) \to A^{q+1}(\mathcal{T})]$, and write $A^q(\mathcal{T}) = \kappa^q \oplus A^q(\mathcal{T})/\kappa^q$. Define $\rho^q : A^q(\mathcal{T}) \twoheadrightarrow H^q(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$ to be the natural projection $\kappa^q \twoheadrightarrow H^q(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$ on κ^q, and trivial on $A^q(\mathcal{T})/\kappa^q$. The map $\omega_{\boldsymbol{\lambda}}^q(\mathcal{T}', \mathcal{T})$ induces an endomorphism

$$\Omega_{\mathcal{C}}^q(\mathcal{T}', \mathcal{T}) : H^q(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}}) \to H^q(A^\bullet(\mathcal{T}), a_{\boldsymbol{\lambda}})$$

determined by the equation $\rho^q \circ \omega_{\boldsymbol{\lambda}}^q(\mathcal{T}', \mathcal{T}) = \Omega_{\mathcal{C}}^q(\mathcal{T}', \mathcal{T}) \circ \rho^q$.

Theorem 7.5 ([CO4]). *Let M be the complement of an arrangement of type $\mathcal{T}$ and let $\mathcal{L}$ be the local system on M defined by weights $\boldsymbol{\lambda}$. Suppose $\mathcal{T}$ covers $\mathcal{T}'$. Then the connection endomorphism $\Omega_A^q(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$ is determined by the equation*

$$\rho^q \circ \omega_{\boldsymbol{\lambda}}^q(\mathcal{T}', \mathcal{T}) = \Omega_A^q(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T})) \circ \rho^q.$$

and hence $\Omega^q_A(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T})) = \Omega^q_{\mathcal{C}}(\mathcal{T}', \mathcal{T})$. $\square$

Now consider the endomorphisms $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$ of the local system cohomology groups $H^q(\mathsf{M}; \mathcal{L})$. Recall from Theorem 4.1 that this cohomology is naturally isomorphic to the cohomology of the Morse theoretic complex $(K^\bullet(\mathcal{A}), \Delta^\bullet)$. As above, let $\varkappa^q = \ker[\Delta^q : K^q(\mathcal{A}) \to K^{q+1}(\mathcal{A})]$, and write $K^q(\mathcal{A}) = \varkappa^q \oplus K^q(\mathcal{A})/\varkappa^q$. Define $\varphi^q : K^q(\mathcal{A}) \twoheadrightarrow H^q(\mathsf{M}; \mathcal{L})$ to be the natural projection $\varkappa^q \twoheadrightarrow H^q(\mathsf{M}; \mathcal{L})$ on $\varkappa^q$, and trivial on $K^q(\mathcal{A})/\varkappa^q$.

Theorem 7.6 ([CO4])**.** *Let* M *be the complement of an arrangement of type* $\mathcal{T}$ *and let* $\mathcal{L}$ *be the local system on* M *defined by weights* $\boldsymbol{\lambda}$*. Suppose* $\mathcal{T}$ *covers* $\mathcal{T}'$*. Then there is an isomorphism* $\tau^q : A^q(\mathcal{T}) \to K^q(\mathcal{A})$ *so that the Gauss-Manin endomorphism* $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$ *in local system cohomology is determined by the equation*

$$\varphi^q \circ \tau^q \circ \omega^q_{\boldsymbol{\lambda}}(\mathcal{T}', \mathcal{T}) = \Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T})) \circ \varphi^q \circ \tau^q. \quad \square$$

8. Spectrum

The eigenvalues of the Gauss-Manin connection satisfy:

Theorem 8.1 ([CO2])**.** *The eigenvalues of the universal Gauss-Manin endomorphism* $\omega^q(\mathcal{T}', \mathcal{T})$ *are integral linear forms in the variables* $y_1, \dots, y_n$*. Thus for any system of weights* $\boldsymbol{\lambda}$*, the eigenvalues of the Gauss-Manin endomorphism in local system cohomology,* $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'), \mathsf{B}(\mathcal{T}))$*, are integral linear combinations of the weights* $\boldsymbol{\lambda}$*.* $\square$

In [CO5] we determined the spectra of these Gauss-Manin endomorphisms. Recall the collection $\mathrm{Dep}(\mathcal{T})$. Here it suffices to work with a smaller collection of dependent sets

$$\mathrm{Dep}(\mathcal{T})^*_q = \{S \in \mathrm{Dep}(\mathcal{T})_q \mid \bigcap_{j \in S} H_j \neq \emptyset\}.$$

Let $\mathrm{Dep}(\mathcal{T})^* = \bigcup_q \mathrm{Dep}(\mathcal{T})^*_q$. If $S \in \mathrm{Dep}(\mathcal{T})^*$, then $\mathrm{codim}(\bigcap_{j \in S} H_j) < |S|$. If $\mathcal{T}'$ is a combinatorial type for which $\mathrm{Dep}(\mathcal{T})^* \subset \mathrm{Dep}(\mathcal{T}')^*$, let $\mathrm{Dep}(\mathcal{T}', \mathcal{T})^* = \mathrm{Dep}(\mathcal{T}')^* \setminus \mathrm{Dep}(\mathcal{T})^*$. If $|S| \geq \ell + 2$, then $S \in \mathrm{Dep}(\mathcal{T})$ but $S \in \mathrm{Dep}(\mathcal{T})^*$ if and only if every subset of S of cardinality $\ell + 1$ is dependent.

Denote the cardinality of S by $s = |S|$. For $1 \leq r \leq \min(\ell, s - 1)$, consider the combinatorial type $\mathcal{T}(S, r)$ defined by

$$T \in \mathrm{Dep}(\mathcal{T}(S, r))^* \iff |T \cap S| \geq r + 1.$$

This type is realized by a pencil of hyperplanes indexed by S with a common subspace of codimension r, together with $n - s$ hyperplanes in general position. Note that for $r = 1$ the hyperplanes in S coincide, so $\mathcal{T}(S, r)$ is a multi-arrangement.

Theorem 8.2 ([CO5])**.** *Let* $\mathcal{T}'$ *be a degeneration of a realizable combinatorial type* $\mathcal{T}$*. For each set* $S_i \in \mathrm{Dep}(\mathcal{T}', \mathcal{T})^*$*, let* r_i *be minimal so that* $\mathrm{Dep}(\mathcal{T}(S_i, r_i))^* \subset \mathrm{Dep}(\mathcal{T}')^*$*. Given the collection* $\{(S_i, r_i)\}$ *there is a unique pair* (S, r) *with* $r = \min\{r_i\}$*,* $\mathrm{Dep}(\mathcal{T}(S, r))^* \subset \mathrm{Dep}(\mathcal{T}')^*$*, and for every pair* (S_i, r_i) *where* $r_i = r$*,* $S_i \subset S$*.* $\square$

Let $\mathcal{T}'$ be a degeneration of $\mathcal{T}$. We call the pair (S, r) which satisfies the conditions of Theorem 8.2 the *principal dependence* of the degeneration. Define

$$\tilde{\omega}^\bullet(S, r) = \sum_{K \in \mathrm{Dep}(\mathcal{T}(S, r))^*} m_K(S, r) \cdot \tilde{\omega}^\bullet_K,$$

where $m_K(S,r)$ is the multiplicity of K in type $\mathcal{T}(S,r)$. We showed in the proof of [CO5, Thm. 5.1] that the endomorphisms $\tilde{\omega}^\bullet(S,r)$ and $\tilde{\omega}^\bullet(\mathcal{T}',\mathcal{T})$ of $\mathsf{A}^\bullet(\mathcal{G})$ induce the same endomorphism in $\mathsf{A}^\bullet(\mathcal{T})$, $\omega^\bullet(S,r) = \omega^\bullet(\mathcal{T}',\mathcal{T})$. It follows from Theorem 7.6 that for all weights $\boldsymbol{\lambda}$, the endomorphism $\omega^\bullet_{\boldsymbol{\lambda}}(S,r)$ induces the Gauss-Manin endomorphism $\Omega^\bullet_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'),\mathsf{B}(\mathcal{T}))$. Write $\lambda_S = \sum_{j\in S}\lambda_j$.

Theorem 8.3 ([CO5]). *Suppose $\mathcal{T}$ covers $\mathcal{T}'$ with principal dependence (S,r). Let $\boldsymbol{\lambda}$ be a collection of weights satisfying $\lambda_S \neq 0$. Then $\tilde{\omega}^q_{\boldsymbol{\lambda}}(S,r) : A^q(\mathcal{G}) \to A^q(\mathcal{G})$, the specialization of $\tilde{\omega}^q(S,r)$ at $\boldsymbol{\lambda}$, is diagonalizable, with eigenvalues 0 and λ_S.*

1. *The 0-eigenspace has dimension*

$$\sum_{p=0}^{r}\binom{s}{p}\binom{n-s}{q-p} - \binom{s-1}{r}\binom{n-s}{q-r}.$$

2. *The λ_S-eigenspace has dimension*

$$\sum_{p=r+1}^{\min(q,s)}\binom{s}{p}\binom{n-s}{q-p} + \binom{s-1}{r}\binom{n-s}{q-r}. \quad \square$$

Our last result was stated in [CO5] only for nonresonant weights but applies in full generality:

Theorem 8.4 ([CO5]). *Suppose $\mathcal{T}$ covers $\mathcal{T}'$ with principal dependence (S,r). Let $\boldsymbol{\lambda}$ be a collection of weights satisfying $\lambda_S \neq 0$. Then the Gauss-Manin endomorphism $\Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{T}'),\mathsf{B}(\mathcal{T}))$ is diagonalizable, with spectrum contained in $\{0,\lambda_S\}$.* $\square$

9. A Selberg arrangement

Let $\mathcal{S}$ be the combinatorial type of the Selberg arrangement $\mathcal{A}$ in $\mathbb{C}^2$ with defining polynomial $Q(\mathcal{A}) = u_1u_2(u_1-1)(u_2-1)(u_1-u_2)$ depicted in Figure 1. See [A1, SV2, Ka] for detailed studies of the Gauss-Manin connections arising in the context of Selberg arrangements.

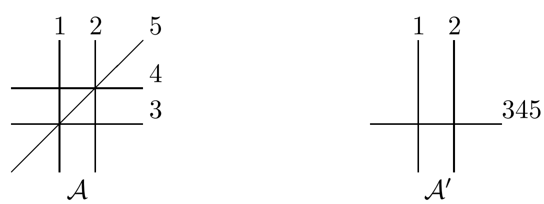

Figure 1. A Selberg Arrangement and One Degeneration

Let $\mathcal{L}$ be the complex rank one local system on the complement M of $\mathcal{A}$ corresponding to the point $\mathbf{t} = (t_1,\dots,t_5) \in (\mathbb{C}^*)^5$. For any such local system on the complement of this arrangement, there is a choice of weights $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_5) \in \mathbb{C}^5$ so that $t_j = \exp(-2\pi\,\mathrm{i}\,\lambda_j)$ for each j, and the local system cohomology $H^*(\mathsf{M};\mathcal{L})$ is isomorphic to the cohomology of the Orlik-Solomon complex $(A^\bullet(\mathcal{S}), a_{\boldsymbol{\lambda}})$. Consequently, if $\mathcal{S}'$ is a degeneration of $\mathcal{S}$, it suffices to compute the Gauss-Manin endomorphism $\Omega^q_A(\mathsf{B}(\mathcal{S}'),\mathsf{B}(\mathcal{S})) = \Omega^q_{\mathcal{L}}(\mathsf{B}(\mathcal{S}'),\mathsf{B}(\mathcal{S})) = \Omega^q_{\mathcal{C}}(\mathcal{S}'\mathcal{S})$.

Let $\mathcal{G}$ be the combinatorial type of a general position arrangement of five lines in $\mathbb{C}^2$. The **nbc** bases for the Orlik-Solomon algebras $A(\mathcal{G})$ and $A(\mathcal{S})$ give rise to bases for the corresponding Aomoto complexes. The Aomoto complex $(\mathsf{A}^\bullet(\mathcal{G}), e_{\mathbf{y}})$ is

(dual to) the rank two truncation of the standard Koszul complex of $\mathbf{y} = y_1, \ldots, y_5$ in the polynomial ring $R = \mathbb{C}[\mathbf{y}]$. The Aomoto complex $(\mathsf{A}^\bullet(\mathcal{S}), a_{\mathbf{y}})$ of the Selberg arrangement is given by

$$\mathsf{A}^0(\mathcal{S}) \xrightarrow{a_{\mathbf{y}}} \mathsf{A}^1(\mathcal{S}) \xrightarrow{a_{\mathbf{y}}} \mathsf{A}^2(\mathcal{S}),$$

where $\mathsf{A}^0(\mathcal{G}) = R$, $\mathsf{A}^1(\mathcal{G}) = R^5$, and $\mathsf{A}^2(\mathcal{G}) = R^6$. Recall that $y_J = \sum_{j \in J} y_j$. The boundary maps of this complex have matrices

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -y_3 & -y_4 & -y_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y_3 & -y_4 & -y_5 \\ y_{15} & 0 & -y_5 & y_2 & 0 & 0 \\ 0 & y_1 & 0 & 0 & y_{25} & -y_5 \\ -y_3 & 0 & y_{13} & 0 & -y_4 & y_{24} \end{bmatrix}.$$

The projection $p : \mathsf{A}^\bullet(\mathcal{G}) \to \mathsf{A}^\bullet(\mathcal{S})$ is given, in the **nbc** bases, by

$$p(e_J) = \begin{cases} 0 & \text{if } J = \{1,2\} \text{ or } J = \{3,4\}, \\ a_{1,5} - a_{1,3} & \text{if } J = \{3,5\}, \\ a_{2,5} - a_{2,4} & \text{if } J = \{4,5\}, \\ a_J & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}'$ denote the combinatorial type of the (multi)-arrangement $\mathcal{A}'$ shown in Figure 1, a codimension one degeneration of $\mathcal{S}$. The principal dependence of this degeneration is (S, r), where $S = 345$ and $r = 1$. The corresponding endomorphism $\tilde{\omega}^\bullet(S, r) : \mathsf{A}^\bullet(\mathcal{G}) \to \mathsf{A}^\bullet(\mathcal{G})$ is given by

$$\begin{aligned} \tilde{\omega}(S, r) = \tilde{\omega}_{34} + \tilde{\omega}_{35} + \tilde{\omega}_{45} + \tilde{\omega}_{134} + \tilde{\omega}_{234} + \tilde{\omega}_{346} + \tilde{\omega}_{135} + \tilde{\omega}_{235} + \tilde{\omega}_{356} \\ + \tilde{\omega}_{145} + \tilde{\omega}_{245} + \tilde{\omega}_{456} + 2\tilde{\omega}_{345}. \end{aligned}$$

The matrices of this chain endomorphism are $\tilde{\omega}^0(S, r) = 0$,

$$\tilde{\omega}^1(S, r) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{45} & -y_4 & -y_5 \\ 0 & 0 & -y_3 & y_{35} & -y_5 \\ 0 & 0 & -y_3 & -y_4 & y_{34} \end{bmatrix},$$

$$\tilde{\omega}^2(S, r) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{45} & -y_4 & -y_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_3 & y_{35} & -y_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -y_3 & -y_4 & y_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_{45} & -y_4 & -y_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_3 & y_{35} & -y_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y_3 & -y_4 & y_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{345} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{345} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{345} \end{bmatrix}.$$

A calculation with the projection $p : \mathsf{A}^\bullet(\mathcal{G}) \to \mathsf{A}^\bullet(\mathcal{S})$ yields the induced endomorphism $\omega^\bullet(\mathcal{S}', \mathcal{S}) = \omega^\bullet(S, r) : \mathsf{A}^\bullet(\mathcal{S}) \to \mathsf{A}^\bullet(\mathcal{S})$, given explicitly by $\omega^0(\mathcal{S}', \mathcal{S}) = 0$,

$\omega^1(\mathcal{S}',\mathcal{S}) = \tilde{\omega}^1(S,r)$, and

$$\omega^2(\mathcal{S}',\mathcal{S}) = \begin{bmatrix} y_{45} & -y_4 & -y_5 & 0 & 0 & 0 \\ -y_3 & y_{35} & -y_5 & 0 & 0 & 0 \\ -y_3 & -y_4 & y_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{45} & -y_4 & -y_5 \\ 0 & 0 & 0 & -y_3 & y_{35} & -y_5 \\ 0 & 0 & 0 & -y_3 & -y_4 & y_{34} \end{bmatrix}.$$

Weights $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ are nonresonant for type $\mathcal{S}$ if

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6,\ \lambda_{135},\ \lambda_{245},\ \lambda_{126},\ \lambda_{346} \notin \mathbb{Z}_{\geq 0},$$

where $\lambda_J = \sum_{j\in J}\lambda_j$ and $\lambda_6 = -\lambda_{[5]}$. The $\beta\mathbf{nbc}$ basis for $H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}}) = H^2(\mathsf{M};\mathcal{L})$ is $\{\eta_{2,4},\eta_{2,5}\}$, where $\eta_{2,j} = (\lambda_2 a_2 + \lambda_4 a_4 + \lambda_5 a_5)\lambda_j a_j$, see [FT]. The projection map $\rho^2 : A^2(\mathcal{S}) \twoheadrightarrow H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}})$ is given by

$$\rho^2(a_{i,j}) = \begin{cases} (\lambda_3\eta_{2,4} - \lambda_1(\eta_{2,4}+\eta_{2,5}))/(\lambda_1\lambda_3\lambda_{135}) & \text{if } \{i,j\}=\{1,3\}, \\ -\eta_{2,4}/(\lambda_1\lambda_4) & \text{if } \{i,j\}=\{1,4\}, \\ (\lambda_{15}\eta_{2,4} + \lambda_1(\eta_{2,4}+\eta_{2,5}))/(\lambda_1\lambda_5\lambda_{135}) & \text{if } \{i,j\}=\{1,5\}, \\ -(\eta_{2,4}+\eta_{2,5})/(\lambda_2\lambda_3) & \text{if } \{i,j\}=\{2,3\}, \\ (\lambda_{24}\eta_{2,4} + \lambda_4\eta_{2,5})/(\lambda_2\lambda_4\lambda_{245}) & \text{if } \{i,j\}=\{2,4\}, \\ (\lambda_5\eta_{2,4} + \lambda_{25}\eta_{2,5})/(\lambda_2\lambda_5\lambda_{245}) & \text{if } \{i,j\}=\{2,5\}. \end{cases}$$

A calculation with the endomorphism $\omega^2_{\boldsymbol{\lambda}}(\mathcal{S}',\mathcal{S}) = \omega^2(\mathcal{S}',\mathcal{S})\big|_{\mathbf{y}\mapsto\boldsymbol{\lambda}}$ and this projection yields

$$\Omega^2_{\mathcal{C}}(\mathcal{S}',\mathcal{S}) = \begin{bmatrix} \lambda_3+\lambda_4+\lambda_5 & 0 \\ 0 & \lambda_3+\lambda_4+\lambda_5 \end{bmatrix}.$$

A collection of weights $\boldsymbol{\lambda}$ is resonant for type $\mathcal{S}$ if $\lambda_1=\lambda_4$, $\lambda_2=\lambda_3$, $\lambda_5=\lambda_6$, and $\lambda_1+\lambda_2+\lambda_5=0$. Let $\boldsymbol{\lambda}$ be a collection of nontrivial, resonant weights. Then $\lambda_1\neq 0$ or $\lambda_2\neq 0$. For such weights, one can check that $a_1-a_2-a_3+a_4 \in A^1(\mathcal{S})$ represents a basis for $H^1(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}})$, and that $\dim H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}}) = 3$. By Theorem 8.4, the spectrum of the Gauss-Manin endomorphism $\Omega^q_{\mathcal{C}}(\mathcal{S}',\mathcal{S})$ is contained in $\{0,\lambda_{345}\}$, provided $\lambda_{345}\neq 0$. However, the resonance conditions above imply that $\lambda_{345}=0$. Accordingly, one can check directly that the endomorphism $\Omega^1_{\mathcal{C}}(\mathcal{S}',\mathcal{S}) : H^1(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}}) \to H^1(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}})$ induced by $\omega^\bullet_{\boldsymbol{\lambda}}(\mathcal{S}',\mathcal{S}) = \omega^\bullet(\mathcal{S}',\mathcal{S})\big|_{\mathbf{y}\mapsto\boldsymbol{\lambda}}$ is trivial. One can also show that, for an appropriate choice of basis for $H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}})$, the projection $A^2(\mathcal{S}) \twoheadrightarrow H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}})$ has matrix

$$\begin{bmatrix} \lambda_1 & -\lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & -\lambda_1 & -\lambda_1 \\ \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_1 & 0 & 0 \end{bmatrix},$$

and that the endomorphism $\Omega^2_{\mathcal{C}}(\mathcal{S}',\mathcal{S}) : H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}}) \to H^2(A^\bullet(\mathcal{S}),a_{\boldsymbol{\lambda}})$ induced by $\omega^\bullet_{\boldsymbol{\lambda}}(\mathcal{S}',\mathcal{S})$ is trivial as well.

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